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Attitude Propagation for a Slewing Angular Rate Vector

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A new finding in attitude propagation for a slewing angular rate vector is derived and validated with a known solution. The method employs an additional coordinate system that slews with the angular rate vector. The method can be applied to general attitude propagation and is ideally suited to propagating the attitude in vehicle computer simulations. The method can also be used to propagate the attitude associated with a sequence of angular increments provided by aerospace vehicle inertial navigation units.

Nomenclature

a, b, c, d	=	polynomial coefficients
D	=	Z-axis drift rate
e_i	=	x, y, and z axis unit vectors
I	=	inertia tensor
N	=	interval index
p	=	roll rate
Q	=	quaternion
q	=	pitch rate
q_i	=	quaternion element
r	=	yaw rate
S	=	auxiliary vector
U	=	transformation matrix
V	=	incremental rotation transformation matrix
v_i	=	auxiliary vector
W_i	=	auxiliary transformation matrix
X_B	=	vector in body coordinates
X_I	=	vector in inertial coordinates
α	=	slew rate of angular rate vector
β	=	slew rate of slewing frame
Δt	=	time increment
ΔU	=	incremental transformation matrix
$\Delta\gamma$	=	auxiliary angle
$\Delta\theta$	=	incremental Euler rotation vector
θ	=	Euler rotation vector
τ	=	applied torque
ϕ	=	rotation vector magnitude for pure coning motion
ω	=	angular rate vector

Introduction

THE attitude of a space vehicle is defined by the relationship between its body reference frame and an inertial reference frame. There are several parameter sets that can be used to define the relationship between the two coordinate frames. The most popular are the Euler rotation vector, the direction cosine transformation matrix (DCM), Euler angles, and quaternions. The Euler rotation vector is aligned with the Euler axis. A single rotation about the Euler rotation axis brings the body frame into alignment with the inertial frame. Thus, the Euler rotation vector involves only three parameters that define the direction and magnitude of the rotation, linking the body to the inertial frame. The DCM contains nine parameters and can transform a vector from the body frame to the inertial frame. Euler angles involve a sequence of three rotations, each about the x, y, or z axis. Quaternions consist of four parameters, three of which represent a vector that is aligned with the Euler rotation vector. The fourth parameter is a redundant scalar. Because the orientation of the body frame with respect to the inertial frame requires a minimum of three parameters, the DCM and quaternions have redundant parameters. Thus, the DCM and the quaternion parameters have to be renormalized periodically to maintain propagation accuracy.

As the attitude of a space vehicle changes, the parameters that specify its orientation with respect to the inertial reference frame also change. The angular rate vector defines the rate of the attitude change. The angular rate vector can also change as a function of time. Euler's equations of motion are typically used to obtain the angular rate vector in space vehicle simulations. Each attitude parameter set can be propagated when the angular rate is known. Thus, there is an associated differential equation driven by the angular rate vector for each parameter set that defines the vehicle attitude.

Attitude propagation is accurate and easy if the angular rate vector points in a fixed direction in space. In this special case, the angular rate vector is also fixed in the vehicle coordinate frame and can be integrated to obtain the Euler rotation vector. In essence, the magnitude of the angular rate vector is integrated, because its direction remains fixed. In fact, the x, y, and z components of the angular rate vector can be integrated separately and then combined into an Euler rotation vector. The order of the infinitesimal rotations can be changed without altering the results, because they are all aligned in the same direction. Once the Euler rotation vector is updated, the other attitude parameters can be readily obtained.

If the angular rate vector changes direction, attitude propagation is more challenging, because the order of the infinitesimal rotations cannot be changed without affecting the results. In these cases, the angular rate vector cannot be integrated directly to yield the Euler rotation vector. One could assume that the angular rate vector remains constant over a short time interval and define a series of incremental angles [1,2]. The incremental rotation angles could then be used to update the attitude. However, there will be an attitude error that grows with propagation time. This is because knowledge of the angular rate vector within each time increment is not known and used in the attitude update. One could reduce the step size to improve accuracy, but the number of steps would have to increase to propagate the attitude over a fixed time duration. A point of diminishing returns would be reached when the increase in accuracy due to smaller step size was outweighed by the increase in numerical error, such as a truncation error.

Instead of reducing step size to improve accuracy, one could numerically evaluate the predominant term in the Euler rotation vector differential equation that is responsible for most of the coning error. This term is referred to as a coning term, and the error is referred to as a coning error, because the error is greatest when the angular rate vector executes a coning or slewing motion. The error can be reduced by assuming that the angular rate vector traces a cone and by

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adjusting the numerical coefficients of a numerical algorithm to minimize the error. Numerous papers were written on the subject of coning correction algorithms to reduce coning error [3–8], which is important for vehicles having strapdown inertial navigation units (INUs).

It is clear that accurate attitude propagation requires not only knowledge of the time-dependent angular rate vector but also how to use that knowledge to propagate the attitude.

The method proposed in this work for propagating the attitude is a complete departure from previous methods. It uses a new finding in rotational kinematics that enables error-free propagation for the case of pure coning motion. The method introduces another coordinate system that is slewing at a constant rate with respect to the vehicle coordinate frame. The angular rate vector is transformed from the vehicle frame to the slewing frame. The attitude propagation is performed in the slewing frame rather than the vehicle body frame. Once the attitude solution is obtained in the slewing frame, it is transformed back to the vehicle body frame. One has the freedom to select the slew rate of the slewing coordinate frame. With the proper choice, the angular rate can be made more benign in the slewing frame than it is in the vehicle frame. For the case of pure coning motion, the angular rate in the slewing frame can be made to point in a fixed direction, thereby permitting error-free propagation.

The method is best illustrated by considering an angular rate vector having a constant magnitude and slewing at a constant rate. If the slew rate of the slewing coordinate frame is the same as the slew rate of the angular rate vector in the vehicle frame, the angular rate has a fixed direction in the slewing frame. This feature permits updating the attitude analytically with no error. Because the attitude is required in the original nonslewing reference frame, a transformation is made back to the original reference frame. This is accomplished analytically using the negative of the slew rate to obtain the Euler rotation vector linking the original frame to the slewing frame. Thus, for a given propagation time, two sequential rotations generate the same attitude propagation as a constant angular rate vector slewing at a constant rate. There is no restriction on the magnitude of the angular rate, the slew rate, or the propagation time. This means that the attitude can be propagated with no error for very long times using two simple rotations.

In realistic cases, the magnitude and the slew rate of the angular rate change. However, over short enough time intervals, the magnitude of the angular rate vector is approximately fixed and its slew rate is also approximately fixed [1]. Thus, the proposed method can be applied to short intervals of time, and the results can be used to update the attitude over the required time span.

In subsequent sections, a short review of traditional attitude parameters and propagation methods is presented. The proposed method is derived mathematically using familiar transformations. The method is validated analytically by applying it to the case of pure coning motion. Because the method requires the slew rate of the angular rate vector, several methods to estimate the slew rate of the angular rate are presented. The method is also applied to a sequence of delta thetas, which are found by integrating the angular rate over short time steps. A change in magnitude and phase enables the method to propagate the attitude with no error for the case of pure coning motion. The method is also applied to a stress case involving time-varying coning motion. A conclusion section summarizes the work.

Attitude Transformation Parameters

The attitude of a vehicle can be specified by a transformation matrix U from the vehicle's coordinate frame to an inertial reference coordinate. If $\mathbf{x_B}$ is a vector in the vehicle body frame, it transforms to the inertial frame by the transformation:

$$\mathbf{x_I} = \mathbf{U}\mathbf{x_B} \tag{1}$$

where $\mathbf{U}$ is the DCM and $\mathbf{x_I}$ is the vector in the inertial frame:

$$\mathbf{U} = \begin{pmatrix} 1+(e_3^2+e_2^2)C & -(e_1e_2C+e_3S) & e_2S-e_1e_3C \\ e_3S-e_1e_2C & 1+(e_1^2+e_3^2)C & -(e_3e_2C+e_1S) \\ -(e_3e_1C+e_2S) & e_1S-e_2e_3C & 1+(e_1^2+e_2^2)C \end{pmatrix} \tag{2}$$

where $\mathbf{e}$ is the unit vector in the direction of an axis of rotation (known as the Euler axis), θ is the angle of rotation about the Euler axis, $C = \cos(\theta) - 1$, and $S = \sin(\theta)$.

Equation (2) can be used to derive the rotation matrix for rotations about the x, y, or z axes. For a rotation about the x axis, $e_1 = 1$, $e_2 = 0$, and $e_3 = 0$. For a rotation about the y axis, $e_1 = 0$, $e_2 = 1$, and $e_3 = 0$. Likewise, for a rotation about the z axis, $e_1 = 0$, $e_2 = 0$, and $e_3 = 1$.

Euler's rotation theorem states that any transformation between two coordinate frames can be represented by a single rotation of angle θ about the Euler rotation axis, as illustrated in Fig. 1. Thus, an Euler rotation vector $\boldsymbol{\theta}$ can be defined to represent an arbitrary transformation, where

$$\boldsymbol{\theta} = \theta\mathbf{e}_1 + \theta\mathbf{e}_2 + \theta\mathbf{e}_3 \tag{3}$$

Given the Euler rotation vector, as in Eq. (3), $\mathbf{U}$ can be obtained via Eq. (2).

The DCM can also be expressed in terms of the quaternion elements:

$$\mathbf{U}= \begin{pmatrix} q_1^2-q_2^2-q_3^2+q_4^2 & 2(q_1q_2-q_3q_4) & 2(q_1q_3+q_2q_4) \\ 2(q_1q_2+q_3q_4) & -q_1^2+q_2^2-q_3^2+q_4^2 & 2(q_2q_3-q_1q_4) \\ 2(q_1q_3-q_2q_4) & 2(q_2q_3+q_1q_4) & -q_1^2-q_2^2+q_3^2+q_4^2 \end{pmatrix} \tag{4}$$

The first three quaternion elements form the vector component of the quaternion. This vector component is aligned with the Euler rotation vector and the fourth quaternion component is a scalar. The quaternion elements are defined by Eqs. (5–9):

$$\mathbf{Q}(\theta) = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ q_4 \end{pmatrix} \tag{5}$$

$$q_1 = e_1 \sin\left(\frac{\theta}{2}\right) \tag{6}$$

$$q_2 = e_2 \sin\left(\frac{\theta}{2}\right) \tag{7}$$

$$q_3 = e_3 \sin\left(\frac{\theta}{2}\right) \tag{8}$$

$$q_4 = \cos\left(\frac{\theta}{2}\right) \tag{9}$$

An attitude transformation can also be represented by a sequence of rotations about the axes of a coordinate system. A typical Euler rotation sequence is yaw, pitch, and roll, for which yaw is about the z axis, pitch is about the y axis, and roll is about the x axis. The total transformation is the product of the individual transformation matrices and is given by

$$\mathbf{U} = \mathbf{U}(\boldsymbol{\theta}_z)\mathbf{U}(\boldsymbol{\theta}_y)\mathbf{U}(\boldsymbol{\theta}_x) \tag{10}$$

Because of the Euler rotation sequences, Euler rotation vectors and quaternions can all be transformed into their corresponding DCMs. The predominant transformation parameter used in this paper is U.

Attitude Propagation

A vehicle's attitude and associated attitude transformation matrix U can change over time. The rate of the attitude change is parameterized in the angular rate vector $\boldsymbol{\omega}(t)$. The attitude time history depends on the functional form of $\boldsymbol{\omega}(t)$, because rotations do not commute. If one alters the time history by interchanging the order of two angular rate increments, the attitude update is altered as well, because

$$\mathbf{U}[\boldsymbol{\omega}(t)\Delta t]U[\boldsymbol{\omega}(t+\Delta t)\Delta t] \neq \mathbf{U}[\boldsymbol{\omega}(t+\Delta t)\Delta t]\mathbf{U}[\boldsymbol{\omega}(t)\Delta t] \tag{11}$$

Therefore, $\mathbf{U}[\boldsymbol{\omega}(t)]$ is a functional that depends on the function $\boldsymbol{\omega}(t)$ and not solely on the integral of $\boldsymbol{\omega}(t)$.

Vehicle attitude can be propagated when the time-dependent angular rate vector is known. Components of the angular rate vector can be used to propagate the attitude parameters that were presented in the previous section. The first-order time-dependent differential equations for quaternion parameters are given by

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right) = \left(\frac{1}{2}\right)\begin{pmatrix} 0 & r & -q & p \\ -r & 0 & p & q \\ q & -p & 0 & r \\ -p & -q & -r & 0 \end{pmatrix}\mathbf{Q} \tag{12}$$

where p, q, and r are the time-dependent roll, pitch, and yaw angular rates about the x, y, and z vehicle body axes, respectively. The elements of the direction cosine matrix can be updated based on the set of differential equations given by

$$\left(\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\right) = \mathbf{U}\begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix} \tag{13}$$

Standard numerical integration techniques can be used to integrate Eqs. (12) and (13). One can integrate the quaternions using Eq. (12) and use them in Eq. (4) to get the DCM, which can be used to transform vectors between the vehicle body and inertial frames. Alternatively, one can integrate the elements of the direction cosine matrix directly using Eq. (13).

Another method to propagate vehicle attitude is combining a series of small rotations associated with a set of small time steps [1]. One assumes that the angular rate vector remains fixed over each small time interval. In a computer simulation, one can use the angular rate at the beginning of the time step, the average angular rate between time steps, or some other numerical procedure to obtain the fixed angular rate over the interval. Strapdown INUs provide the incremental rotations in the form of delta thetas. Each incremental rotation is the product of the angular rate and the associated Δt. Thus, attitude propagation is achieved by a sequence of incremental rotations about the respective Euler axes. The incremental rotation vector and the transformation matrix are given by

$$\boldsymbol{\Delta\theta} = \begin{pmatrix} \Delta\theta_x \\ \Delta\theta_y \\ \Delta\theta_z \end{pmatrix} = \begin{pmatrix} p\Delta t \\ q\Delta t \\ r\Delta t \end{pmatrix} \tag{14}$$

$$\mathbf{V} = \begin{pmatrix} 0 & -\Delta\theta_z & \Delta\theta_y \\ \Delta\theta_z & 0 & -\Delta\theta_x \\ -\Delta\theta_y & \Delta\theta_x & 0 \end{pmatrix} \tag{15}$$

The updated transformation matrix is given by

$$\mathbf{U}(t+\Delta t) = \mathbf{U}(t) + \mathbf{U}(t)\mathbf{V} \tag{16}$$

The associated updated quaternion is given by

$$\mathbf{Q}(t+\Delta t) = \mathbf{Q}(t) + \frac{1}{2}\begin{pmatrix} 0 & \Delta\theta_z & -\Delta\theta_y & \Delta\theta_x \\ -\Delta\theta_z & 0 & \Delta\theta_x & \Delta\theta_y \\ \Delta\theta_y & -\Delta\theta_x & 0 & \Delta\theta_z \\ -\Delta\theta_x & -\Delta\theta_y & -\Delta\theta_z & 0 \end{pmatrix}\mathbf{Q}(t) \tag{17}$$

This attitude propagation procedure was enhanced to improve attitude propagation accuracy. The differential equation for the Euler rotation vector $\boldsymbol{\theta}$ involves the angular rate vector $\boldsymbol{\omega}$ and is given by [8]:

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} + \frac{1}{2}(\boldsymbol{\theta}\times\boldsymbol{\omega}) + \frac{1}{\theta^2}\left(1 - \frac{\theta\sin(\theta)}{2[1-\cos(\theta)]}\right)\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\omega}) \tag{18}$$

Equation (14) is equivalent to Eq. (17) integrated over a short time interval and without the last two terms on the right-hand side. The last two terms on the right-hand side of Eq. (18) account for the change in direction of the angular rate vector. Note that if the angular rate vector remains in a fixed direction, $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ are aligned and

$$\boldsymbol{\theta}\times\boldsymbol{\omega} = 0 \tag{19}$$

In this case, the last two terms in Eq. (18) are zero. These two terms illustrate the noncommutative nature of the rotations, because the functional form of $\boldsymbol{\omega}(t)$ changes $\boldsymbol{\theta}$ and the resulting vehicle attitude. Much work has been done on developing numerical methods to integrate Eq. (18) and are referred to as coning correction algorithms [3–7]. As will be shown, the proposed attitude propagation method does not involve coning correction algorithms.

Attitude Propagation for a Nonslewing Angular Rate Vector

If an angular rate has a fixed direction, it can be integrated directly to determine the time-dependent Euler rotation angle, which determines the transformation via Eq. (2). In this case, the angular rate remains aligned with the Euler axis and the Euler rotation vector:

$$\theta(t) = \int_0^t \omega(t')\,\mathrm{d}t' \tag{20}$$

This is equivalent to integrating Eq. (18) without the last two terms and obtaining the equivalent Euler rotation vector. Thus, only a simple integration of a one-dimensional integral is needed to obtain the attitude solution. If ω is constant, Eq. (18) can be analytically integrated to obtain the closed-form solution for the Euler rotation vector:

$$\boldsymbol{\theta}(t) = \mathbf{e}\theta(t) = \boldsymbol{\omega}t \tag{21}$$

The associated DCM is obtained using Eq. (2):

$$\mathrm{DCM} = \mathbf{U}[\boldsymbol{\theta}(t)] \tag{22}$$

Proposed Method

This simplicity of the attitude solution for an angular rate vector that is not slewing provided the motivation for the proposed method. Consider a vehicle angular rate vector of constant magnitude that is slewing at a constant angular rate. If the angular rate vector in the vehicle frame is slewing with an angular rate $\boldsymbol{\alpha}$, its time dependence is given by

$$\boldsymbol{\omega}(t) = U(\boldsymbol{\alpha}t)\boldsymbol{\omega}(0) \tag{23}$$

Another coordinate frame is created that is slewing with an angular rate given by $\boldsymbol{\alpha}$. Any vector in the vehicle frame can be transformed to the slewing frame via the coordinate rotation given by

$$\mathbf{X}_s = \mathbf{U}^{-1}(\boldsymbol{\alpha}t)\mathbf{X} = \mathbf{U}(-\boldsymbol{\alpha}t)\mathbf{X} \tag{24}$$

Using Eqs. (23) and (24), one obtains the angular rate of the vehicle frame when it is transformed to the slewing frame:

$$\boldsymbol{\omega}_v = \mathbf{U}(-\boldsymbol{\alpha}t)\mathbf{U}(\boldsymbol{\alpha}t)\boldsymbol{\omega}(0) = \boldsymbol{\omega}(0) \tag{25}$$

Therefore, in the slewing frame, the vehicle's angular rate vector remains fixed at its value at time $t = 0$. However, the angular rate in Eq. (25) is not the angular rate of the slewing frame itself, because the slewing frame has an additional angular rate with respect to the vehicle frame, given by $\boldsymbol{\alpha}$. Thus, the slewing frame has an angular rate, with respect to the inertial frame, given by

$$\boldsymbol{\omega}_s = \boldsymbol{\omega}(0) + \boldsymbol{\alpha} \tag{26}$$

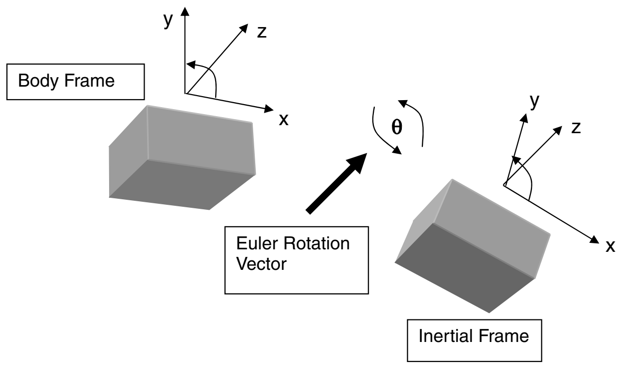

Fig. 1 Euler rotation vector with inertial and body coordinate frames.

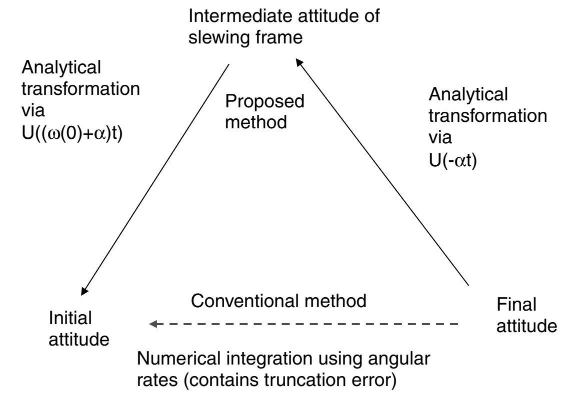

Fig. 2 Illustration of proposed and conventional attitude propagation methods.

Therefore, a vector in the slewing frame transforms to the inertial frame via

$$\mathbf{X}_I = \mathbf{U}[\boldsymbol{\omega}(0)t + \boldsymbol{\alpha}t]\mathbf{X}_S \tag{27}$$

Using Eqs. (24) and (27), a vector in the vehicle frame can be transformed to the inertial frame by first transforming it to the slewing frame and then transforming it to the inertial frame. The two sequential transformations are given by

$$\mathbf{X}_I = \mathbf{U}[\boldsymbol{\omega}(0)t + \boldsymbol{\alpha}t]U(-\boldsymbol{\alpha}t)\mathbf{X} \tag{28}$$

If a vehicle has an angular rate of constant magnitude that is slewing at a constant rate with respect to the vehicle frame, the attitude can be propagated analytically with no error using two simple rotations, as given in Eq. (28).

Equation (28) is exact and is more computationally efficient than the standard attitude propagation methods represented in Eqs. (12) and (13). Equation (28) holds for arbitrarily long time periods, as long as the magnitude of the angular rate is constant and the slew rate $\boldsymbol{\alpha}$ is constant. Figure 2 illustrates the proposed method, which involves analytical transformations rather than numerical integration used by conventional methods.

The use of the slewing coordinate system may have value even if the angular rate of the vehicle is not slewing at a constant rate. Consider the attitude propagation problem posed by a vehicle with a time-varying angular rate vector, given by $\boldsymbol{\omega}(t)$. It may be preferable to solve the problem in the slewing frame, which has an angular rate given by

$$\boldsymbol{\omega}_s(t) = \mathbf{U}[-\boldsymbol{\beta}t]\boldsymbol{\omega}(t) + \boldsymbol{\beta} \tag{29}$$

$$\mathrm{DCM} = \begin{pmatrix} \cos(\phi) + [1-\cos(\phi)]\sin^2(\alpha t) & 0.5[1-\cos(\phi)]\sin(2\alpha t) & \sin(\phi)\cos(\alpha t) \\ 0.5[1-\cos(\phi)]\sin(2\alpha t) & \cos(\phi) + [1-\cos(\phi)]\cos^2(\alpha t) & -\sin(\phi)\sin(\alpha t) \\ -\sin(\phi)\cos(\alpha t) & \sin(\phi)\sin(\alpha t) & \cos(\phi) \end{pmatrix} \tag{34}$$

where $\boldsymbol{\beta}$ is the slew rate of the slewing frame. The attitude transformation from the slewing frame to the inertial frame is given by

$$\mathbf{X}_I = \mathbf{U}[\boldsymbol{\theta}_s(t)]\mathbf{X}_s \tag{30}$$

where $\boldsymbol{\theta}_s$ is the Euler rotation vector associated with $\boldsymbol{\omega}_s$. If the angular rate of the slewing frame undergoes a smaller variation than the angular rate in the vehicle frame, the resulting attitude computation may be more accurate.[†] The attitude solution is obtained in the slewing frame as $\mathbf{U}[\boldsymbol{\theta}_s(t)]$. When needed, one calculates the transformation from the vehicle frame to the inertial frame easily, because it amounts to an additional simple rotation of $-\boldsymbol{\beta}t$, as shown in Eq. (31).

$$\mathbf{X}_I = \mathbf{U}[\boldsymbol{\theta}_s(t)]\mathbf{U}(-\boldsymbol{\beta}t)\mathbf{X} \tag{31}$$

Validation of Method

The method was tested using a published example involving pure coning motion [8], represented by Eqs. (32–34). Assume that the vehicle has an Euler rotation vector given by

$$\boldsymbol{\theta}(t) = \begin{pmatrix} \phi\sin(\alpha t) \\ \phi\cos(\alpha t) \\ 0 \end{pmatrix} \tag{32}$$

The corresponding angular velocity vector is given by

$$\boldsymbol{\omega}(t) = \begin{pmatrix} \alpha\sin(\phi)\cos(\alpha t) \\ -\alpha\sin(\phi)\sin(\alpha t) \\ \alpha[1-\cos(\phi)] \end{pmatrix} \tag{33}$$

The associated DCM is found by using Eq. (32) in Eq. (2) and is given by

To make the notation compatible with the notation in this paper, $\boldsymbol{\alpha}$ is used instead of $\boldsymbol{\omega}$, which appears in the published version of Eqs. (32–34). This helps to avoid confusion, because $\boldsymbol{\omega}$ is associated with the angular rate rather than the slew rate or the coning rate.

The proposed method was applied to this known solution by propagating the vehicle attitude based on initial conditions at $t = 0$. The vehicle's attitude was initialized by applying the Euler rotation vector, given by Eq. (32) at $t = 0$:

$$\boldsymbol{\theta}(0) = \begin{pmatrix} 0 \\ \phi \\ 0 \end{pmatrix} \tag{35}$$

[†]This variation of the proposed method was suggested by F. L. Markley in a private communication.

The initial angular rate was provided by Eq. (33), with $t = 0$:

$$\boldsymbol{\omega}(0) = \begin{pmatrix} \alpha\sin(\phi) \\ 0 \\ \alpha[1-\cos(\phi)] \end{pmatrix} \tag{36}$$

The angular rate given by Eq. (36) is constant in magnitude and cones about the z axis with a rate given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 \\ 0 \\ -\alpha \end{pmatrix} \tag{37}$$

Using Eq. (28), with the initial attitude offset, the time-dependent transformation from the vehicle frame to the initial offset frame is given by

$$\mathbf{U}_s(t) = \mathbf{U}[\boldsymbol{\theta}(0)]\mathbf{U}[(\boldsymbol{\omega}(0)+\boldsymbol{\alpha})t]\mathbf{U}(-\boldsymbol{\alpha}t) \tag{38}$$

Equation (2) is used to obtain the explicit matrices in Eq. (38). Using Eq. (35) in Eq. (2) yields the transformation from the inertial frame to the initial attitude offset frame:

$$\mathbf{U}(\boldsymbol{\theta}) = \begin{pmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{pmatrix} \tag{39}$$

Using Eqs. (2), (36), and (37) yields the transformation from the slewing frame to the inertial frame:

$$\mathbf{U}[(\boldsymbol{\omega}(0)+\boldsymbol{\alpha})t] = \begin{pmatrix} \sin^2(\phi)+\cos^2(\phi)\cos(\alpha t) & \cos(\phi)\sin(\alpha t) & \cos(\phi)\sin(\phi)[\cos(\alpha t)-1] \\ -\cos(\phi)\sin(\alpha t) & \cos(\alpha t) & -\sin(\phi)\sin(\alpha t) \\ \cos(\phi)\sin(\phi)[\cos(\alpha t)-1] & \sin(\phi)\sin(\alpha t) & \cos^2(\phi)+\sin^2(\phi)\cos(\alpha t) \end{pmatrix} \tag{40}$$

The transformation from the vehicle frame to the slewing frame is obtained using Eqs. (2), (37), and (38):

$$\mathbf{U}(-\boldsymbol{\alpha}t) = \begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) & 0 \\ \sin(\alpha t) & \cos(\alpha t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{41}$$

Using Eqs. (39–41) in Eq. (38) and carrying out the multiplications yields:

$$\mathbf{U}_s(t) = \begin{pmatrix} \cos(\phi)+[1-\cos(\phi)]\sin^2(\alpha t) & 0.5[1-\cos(\phi)]\sin(2\alpha t) & \sin(\phi)\cos(\alpha t) \\ 0.5[1-\cos(\phi)]\sin(2\alpha t) & \cos(\phi)+[1-\cos(\phi)]\cos^2(\alpha t) & -\sin(\phi)\sin(\alpha t) \\ -\sin(\phi)\cos(\alpha t) & \sin(\phi)\sin(\alpha t) & \cos(\phi) \end{pmatrix} \tag{42}$$

Equation (42) is identical to Eq. (34) and shows that the proposed method can propagate attitude analytically with no error for the case of pure coning motion. Notice that only the initial values of the attitude offset, the angular rate, and the slew rate were used to obtain Eq. (42), indicating that the attitude was analytically propagated from the initial conditions.

Simulation Example

The method can be applied to the problem of simulating the attitude of an axisymmetric space vehicle having spin and transverse moments of inertia given by I_s and I_t, respectively. Assuming that no torques are acting on the vehicle, the coning frequency or the slew rate of the angular rate vector in the vehicle frame is given by

$$\boldsymbol{\alpha} = \boldsymbol{\omega}_{\mathbf{s}}(0)\left(\frac{I_s}{I_t}-1\right) \tag{43}$$

where $\boldsymbol{\omega}_{\mathbf{s}}(\mathbf{0})$ is the component of the angular rate vector aligned with the spin axis at $t = 0$. Notice that the slew rate is also aligned with the spin axis. The attitude can be propagated analytically using Eq. (28), because both the magnitude of the angular rate vector and its slew rate in the vehicle frame are constant. The attitude can be propagated by

$$\mathbf{U} = \mathbf{U}\{[\boldsymbol{\omega}(0)+\boldsymbol{\alpha}]t\}\mathbf{U}(-\boldsymbol{\alpha}t) \tag{44}$$

Note that, because $\boldsymbol{\omega}(\mathbf{0})$ has a transverse component, it is not aligned with $\boldsymbol{\alpha}$. Equation (44) indicates that it takes only two simple rotations to propagate the attitude of an axisymmetric space vehicle undergoing torque-free rotational motion.

Application of Proposed Method to General Attitude Motion

Because angular rate vectors for typical vehicles are not usually constant in magnitude and do not usually slew at constant rates for extended periods of time, Eq. (28) can only be applied for small increments of time. For short enough time increments, the angular rate magnitude is approximately constant, and its associated slew rate is approximately constant. Both the angular rate and the slew rate can be obtained using Euler's equations of motion. This makes the proposed method ideally suited for computing the attitude in space vehicle simulations. Attitude can be propagated over each time increment analytically using Eq. (28). Because the time history of the angular rate vector has already been included via the slew rate, there is no coning error using this method.

Once the slew rate for each updated time interval is computed and used in Eq. (28), the total transformation up to the current time can be found as

$$\mathbf{U}_{\mathbf{t}} = \Delta\mathbf{U}_{\mathbf{1}},\ \Delta\mathbf{U}_{\mathbf{2}},\ \Delta\mathbf{U}_{\mathbf{3}},\ \Delta\mathbf{U}_{\mathbf{4}},\ \ldots,\ \Delta\mathbf{U}_{\mathbf{N}} \tag{45}$$

where the increments are numbered sequentially. Each increment involves two simple analytical rotations. If the slew rate of the angular rate vector is not known, it must be computed. If the slew rate of the angular rate vector does not change over the N time increments, the attitude can be transformed using only two rotations given by

$$\mathbf{U}_{t_N} = \mathbf{U}\{[\boldsymbol{\omega}(0)+\boldsymbol{\alpha}]t_N\}\mathbf{U}(-\boldsymbol{\alpha}t_N) \tag{46}$$

where t_N is the total time duration for the N time increments.

There are numerous methods to compute the slew rate of the angular rate vector. A few methods are provided.

Cross Product Method

Assume that a time sequence of the angular rate vectors is available. A moving window of two consecutive angular rate vectors is processed to obtain a sequence of slew rates. If the time step between the two successive angular rate vectors is small, so that the angular rate vector does not change too much, Eq. (28) can be used. The slew rate can be obtained by a vector cross-product operation, which is used to obtain the angle between the two angular rate vectors:

$$\mathbf{S}_1 = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 \tag{47}$$

$$\Delta\gamma = \sin^{-1}\left(\frac{|\mathbf{S}_1|}{|\boldsymbol{\omega}_1||\boldsymbol{\omega}_2|}\right) \tag{48}$$

The associated slew angular rate vector is given by

$$\boldsymbol{\alpha}_1 = \frac{\mathbf{S}_1 \Delta\gamma}{|\mathbf{S}_1|(\Delta t)} \tag{49}$$

This method assumes that the slew rate vector is normal to the plane containing the two angular rate vectors, which may not be true. Therefore, this method yields only an approximate slew rate. However, for sufficiently small time steps, it can yield accurate results for attitude propagation. The associated transformation matrix is obtained, based on Eq. (28):

$$\Delta\mathbf{U}_1 = \mathbf{U}[(\boldsymbol{\omega}_1 + \boldsymbol{\alpha}_1)\Delta t]\mathbf{U}[(-\boldsymbol{\alpha}_1)\Delta t] \tag{50}$$

The moving window is advanced one time step and the new set of two angular rate vectors are processed in a similar fashion, using Eqs. (47–50) to obtain the next incremental transformation matrix $\boldsymbol{\Delta}\mathbf{U_2}$. The process is repeated until all the angular rate vectors in the sequence, up to the current time, are processed. The resultant transformation matrix U_t is found by multiplying the N incremental transformations in Eq. (45).

Coordinate Transformation Method

A higher-fidelity method involves using three consecutive angular rate vectors. The first two angular rate vectors can be used to define two vectors that define a coordinate transformation W_1:

$$\mathbf{v}_1 = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 \tag{51}$$

$$\mathbf{v}_2 = \boldsymbol{\omega}_1 \times \mathbf{v}_1 \tag{52}$$

After unitizing $\boldsymbol{\omega}_1$, $\mathbf{v}_1$, and $\mathbf{v}_2$, the transformation $\mathbf{W}_1$ can be defined as

$$\mathbf{W}_1 = (\boldsymbol{\omega}_1 \quad \mathbf{v}_1 \quad \mathbf{v}_2) \tag{53}$$

This process is repeated with the second and third angular rate vectors to obtain $\mathbf{W}_2$:

$$\mathbf{v}_3 = \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_3 \tag{54}$$

$$\mathbf{v}_4 = \boldsymbol{\omega}_2 \times \mathbf{v}_3 \tag{55}$$

$$\mathbf{W}_2 = (\boldsymbol{\omega}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4) \tag{56}$$

The transformation between $\mathbf{W}_1$ and $\mathbf{W}_2$ is $\mathbf{U}(\boldsymbol{\alpha}_1\Delta t)$ and is obtained using Eq. (57):

$$\mathbf{U}(\boldsymbol{\alpha}_1\Delta t) = \mathbf{W}_2\mathbf{W}_1^{-1} \tag{57}$$

The slew rate α_1 is obtained from Eq. (57) using standard methods and is used in Eq. (50) to update the attitude over the associated time interval. The derived slew rate is unique and is assumed to be constant over the associated time interval.

Once again, the moving window is advanced one time step, and the next three angular rate vectors are processed to obtain the next slew rate α_2. The process is repeated for the subsequent time steps to obtain the total transformation, given by Eq. (45).

Curve Fit Method

One can use another algorithm to obtain the slew rate of the angular rate vector that involves a sequence of several angular rate vectors over an attitude update interval. The components of the angular rate vector can be fit to a low-order polynomial given by

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k} \tag{58}$$

$$\omega_i = a_i + b_i t + c_i t^2 + d_i t^3 \tag{59}$$

where i ranges over x, y, and z. The first and second time derivatives of the angular rate vector are obtained from Eq. (59) as

$$\dot{\omega}_i = b_i + 2c_i t + 3d_i t^2 \tag{60}$$

$$\ddot{\omega}_i = 2c_i + 6d_i t \tag{61}$$

The rate at which the angular rate vector is slewing is given by Eq. (62) or Eq. (63):

$$\boldsymbol{\alpha} = \frac{\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}} \tag{62}$$

$$\boldsymbol{\alpha} = \frac{\dot{\boldsymbol{\omega}} \times \ddot{\boldsymbol{\omega}}}{\dot{\boldsymbol{\omega}} \cdot \dot{\boldsymbol{\omega}}} \tag{63}$$

where the numerator of the right-hand side involves the vector cross product and the denominator involves the dot product. Equation (62) is not as accurate as Eq. (63), because it assumes that the slew rate is directed normal to both ω and $\dot{\omega}$, which may not be true. Equations (58–63) are applied for each attitude update interval to obtain the associated slew rate and the transformation matrix given by

$$\Delta\mathbf{U}_j = \mathbf{U}[(\boldsymbol{\omega}_j + \boldsymbol{\alpha}_j)\Delta t]\mathbf{U}[(-\boldsymbol{\alpha}_j)\Delta t] \tag{64}$$

where j is the sequential index for the update interval. The total transformation is given by Eq. (45), which is a product of the sequential transformations give by Eq. (64).

Euler Equation Method

Propagating the vehicle attitude in computer simulations can be accomplished using Euler's equations to obtain the angular rate vector and its derivatives. Euler's equations can be expressed as

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \mathbf{I}^{-1}[(\mathbf{I}\boldsymbol{\omega}) \times \boldsymbol{\omega}] + \mathbf{I}^{-1}\boldsymbol{\tau} \tag{65}$$

where $\mathbf{I}$ is the inertia tensor expressed in the body frame and $\boldsymbol{\tau}$ is the applied torque to the body. Euler's equations provide $\boldsymbol{\omega}$ and $\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t$, which can be used in Eq. (62) to obtain $\boldsymbol{\alpha}$. Alternatively, the derivative of Euler's equation provides $\mathrm{d}^2\boldsymbol{\omega}/\mathrm{d}t^2$, which can be used in Eq. (63) to obtain $\boldsymbol{\alpha}$. As mentioned earlier, Eq. (63) is more accurate than Eq. (62).

These particular numerical implementations to obtain slew rates are presented for illustration purposes. Other implementations can be developed but are not presented here, because they are tangential to the main development of this work.

Application to Strapdown Inertial Navigation Units

Strapdown INUs are used on flight vehicles to determine vehicle position, velocity, and attitude. The attitude rate sensors or gyros generate a sequence of angular increments $\boldsymbol{\Delta\theta}$, which are integrals of the angular rate over the associated time increment. Thus, each $\boldsymbol{\Delta\theta}$ can be thought of as the average of the angular rate vector over the time increment multiplied by Δt. The angular increments can be used directly in Eqs. (2) or (4) to propagate vehicle attitude. The smaller the time increment, the smaller the propagation error. In particular, there is a drift error about the z axis for pure coning motion that degrades the attitude propagation accuracy. The error arises because angular rotations do not commute, making the order of the rotations important. Let $\boldsymbol{\omega}(t)$ be the time-dependent angular rate. The angular increment $\boldsymbol{\Delta\theta}$, from t to $t+\Delta t$, is given by the integral of the angular rate:

$$\boldsymbol{\Delta\,\theta} = \int_t^{t+\Delta t} \boldsymbol{\omega}(\tau)\,\mathrm{d}\tau \tag{66}$$

There are many different $\boldsymbol{\omega}(\tau)$ that could be used in Eq. (66) that would result in the same $\boldsymbol{\Delta\theta}$ but generate different vehicle attitudes. It is lack of knowledge and use of the angular rate profile within each $\boldsymbol{\Delta\theta}$ increment that results in the attitude propagation error. For pure coning motion, the attitude drift rate error can be computed analytically by integrating the second term in Eq. (18) over time Δt [9]:

$$D = \left(\frac{\sin^2(\phi)\alpha}{2}\right)\left(1 - \frac{\sin(\alpha\Delta t)}{\alpha\Delta t}\right) \tag{67}$$

The drift rate quantifies our lack of knowledge of the angular rate over the Δt interval. Coning correction algorithms gain some knowledge of the angular rate profile within each $\boldsymbol{\Delta\theta}$ by using vector cross products of $\boldsymbol{\Delta\theta}$ pairs to obtain a measure of the coning rate. Numerical coefficients in coning correction algorithms are selected, such that the error is reduced for the case of pure coning motion. For example, a coning algorithm [9] using two cross products, involving three sequential data samples for each Δt, reduces the drift rate from that of Eq. (67) to

$$D_c = \frac{\sin^2(\phi)\alpha^7(\Delta t)^6}{204,120} \tag{68}$$

Thus, each coning correction algorithm is essentially a numerical method to evaluate the second term in Eq. (18) using the assumption of pure coning motion. Coning correction works very well for general motion of the angular rate vector as well as pure coning motion.

The method of attitude propagation proposed in this work uses the instantaneous angular rate vector at the beginning of each $\boldsymbol{\Delta\theta}$ rather than $\boldsymbol{\Delta\theta}$ itself. Using a sequence of $\boldsymbol{\Delta\theta}$, it is possible to determine the slew rate of the angular rate vector. Once the slew rate is found, it can be used with $\boldsymbol{\Delta\theta}$ to obtain the angular rate vector at the beginning of each $\boldsymbol{\Delta\theta}$ increment. When this is accomplished, the vehicle attitude can be propagated very accurately. In the case of pure coning motion, vehicle attitude can be propagated without error.

The first step is to estimate the slew rate of the angular rate vector. This can be accomplished using the methods presented in the previous section with a sequence of $\boldsymbol{\Delta\theta}$ replacing the sequence of instantaneous angular rate vectors. This uses the common assumption that the slew rate of the angular rate vector remains constant over several time increments.

The component of the angular rate that is parallel to the slew rate $\boldsymbol{\alpha}$ remains constant during the time interval associated with $\boldsymbol{\Delta\theta}$. The component of the angular rate that is normal to $\boldsymbol{\alpha}$ rotates about $\boldsymbol{\alpha}$ by an angle given by $\alpha\Delta t$, as shown in Fig. 3. The components of $\boldsymbol{\Delta\theta}$ that are normal and parallel to $\boldsymbol{\alpha}$ are given by Eqs. (69) and (70), respectively:

$$\boldsymbol{\Delta\,\theta_p} = \left(\frac{\boldsymbol{\Delta\theta}\cdot\boldsymbol{\alpha}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}\right)\boldsymbol{\alpha} \tag{69}$$

$$\boldsymbol{\Delta\,\theta_n} = \boldsymbol{\Delta\theta} - \boldsymbol{\Delta\theta_p} \tag{70}$$

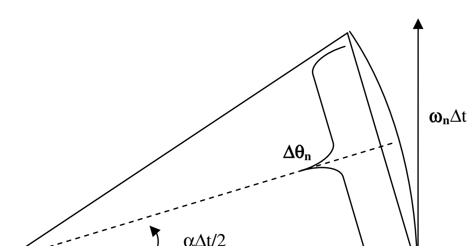

Fig. 3 Relationship between $\boldsymbol{\Delta\theta_n}$ and the normal component of the instantaneous angular rate $\boldsymbol{\omega_n}$.

The arc length illustrated in Fig. 3 is the magnitude of the angular rate vector multiplied by the time increment. Therefore, one can obtain the magnitude of the normal component of the angular rate vector through its relationship with $\boldsymbol{\Delta\theta_n}$:

$$|\boldsymbol{\omega_n}| = \frac{|\boldsymbol{\Delta\theta_n}|\alpha}{2\sin(\alpha\Delta t/2)} \tag{71}$$

The direction of the angular rate vector at the beginning of the time interval is the same as the direction of $\boldsymbol{\Delta\theta_n}$ when rotated by the Euler rotation vector $(-\boldsymbol{\alpha}\Delta t/2)$. Equation (72) provides the instantaneous angular rate at the beginning of an angular increment:

$$\boldsymbol{\omega} = \frac{\Delta\boldsymbol{\theta_p}}{\Delta t} + |\boldsymbol{\omega_n}|\cos\left(\frac{-\alpha\Delta t}{2}\right) + |\boldsymbol{\omega_n}|\sin\left(\frac{-\alpha\Delta t}{2}\right)\left(\frac{\boldsymbol{\alpha}\times\Delta\boldsymbol{\theta_n}}{|\boldsymbol{\alpha}\times\Delta\boldsymbol{\theta_n}|}\right) \tag{72}$$

Using the angular rate given by Eq. (72) and the derived slew rate $\boldsymbol{\alpha}$, the attitude can be propagated with zero error for the case of pure coning motion, using Eq. (28).

Numerical Results

Pure Coning Motion

A computer simulation was developed to generate the angular rate and attitude for pure coning motion, using Eqs. (32) and (33). The numerical values were then processed to determine the slew rate, using the coordinate transformation method. The attitude was computed using Eq. (28) and found to agree with the results, using Eq. (34). Because this case involves pure coning motion, the attitude was propagated using only two rotations, regardless of the propagating time. This example merely demonstrates the validity of the slew rate estimation method and the simulation code, because Eq. (42) was validated analytically earlier.

A simulation was developed to generate delta thetas and the corresponding vehicle attitude for the case of pure coning motion. Equation (33) was used in Eq. (66) to obtain the delta thetas analytically, as given by

$$\int_{t_i}^{t_i+\Delta t} \omega(t)\,\mathrm{d}t = \begin{pmatrix} \sin(\phi)\{\sin[\alpha(t_i+\Delta t)] - \sin(\alpha t_i)\} \\ \sin(\phi)\{\cos[\alpha(t_i+\Delta t)] - \cos(\alpha t_i)\} \\ \alpha[1-\cos(\phi)]\Delta t \end{pmatrix} \tag{73}$$

Equation (73) was used to compute a delta theta for each time increment. The corresponding vehicle attitude for each delta theta was obtained using Eq. (32).

Another simulation was developed to estimate the slew rate of the delta thetas and process the data using Eq. (72) to compute the angular rate vector at the beginning of each time increment. Equation (28) was then used to propagate the attitude.

Table 1 illustrates the results for a pure coning example, given by Eqs. (32) and (73), with a frequency of 50 Hz and a tilt angle ϕ of 2 deg. Drift rates associated with the uncompensated case, a coning correction case, and the proposed method were computed using Eqs. (67), (68), and (28), respectively. Theoretically, the proposed method has no error. The error shown in Table 1 is a numerical error due to generating the attitude and the angular rate data, using Eqs. (32) and (73), finding the slew rate using the coordinate transformation method, deriving the angular rate vectors from the delta thetas using Eq. (72), propagating the attitude, and determining the error based on the true attitude given by Eq. (32). It should be noted that higher-order coning compensation algorithms that use

Table 1 Pure coning example results

Measurement frequency, Hz	Update frequency, Hz	Uncompensated drift rate, deg /h	Coning correction drift rate, deg /h	Proposed method drift rate, deg /h
300	100	39,478	372	-2.28×10^{-9}
450	150	23,154	32.6	-6.9×10^{-9}
600	200	14,346	5.81	-8.02×10^{-9}
750	250	9,600	1.52	4.94×10^{-9}
900	300	6,830	0.51	-1.17×10^{-8}
1,050	350	5,092	0.20	1.97×10^{-8}
1,200	400	3,935	0.09	7.81×10^{-9}
1,350	450	3,130	0.0448	-8.5×10^{-9}
1,500	500	2,547	2.38×10^{-2}	-1.73×10^{-8}
2,250	750	1,144	2.09×10^{-3}	1.04×10^{-8}
3,000	1,000	646	3.72×10^{-4}	-6.67×10^{-8}

more that three data samples per update have been developed [3–7] and result in greater accuracy. The three-sample case shown in Table 1 is for illustration purposes.

Results in Table 1 show that the proposed method can be used to propagate vehicle attitude with nearly no error using delta thetas, although more computational effort is required than the published coning correction algorithms.

Time-Varying Coning Motion

A numerical example using time-varying coning motion was examined to demonstrate that the proposed method can work for cases other than pure coning motion. Consider the vehicle attitude defined by the Euler rotation vector given by

$$\boldsymbol{\theta}(t) = \begin{pmatrix} \phi\cos(\nu t)\sin(\alpha t) \\ \phi\cos(\nu t)\cos(\alpha t) \\ 0 \end{pmatrix} \tag{74}$$

This represents coning motion with an amplitude oscillating at the angular frequency ν. The corresponding angular rate can be found using the equation given by [8], where $\boldsymbol{\theta}$ is the Euler rotation vector defining the vehicle attitude:

$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} - \frac{[1-\cos(\theta)]}{\theta^2}(\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}) + \frac{[\theta-\sin(\theta)]}{\theta^3}[\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}})] \tag{75}$$

Using Eq. (74) in Eq. (75), one finds the angular rate given by

$$\boldsymbol{\omega} = \begin{pmatrix} -\phi\nu\sin(\nu t)\sin(\alpha t) + \alpha\sin[\phi\cos(\nu t)]\cos(\alpha t) \\ -\phi\nu\sin(\nu t)\cos(\alpha t) - \alpha\sin[\phi\cos(\nu t)]\sin(\alpha t) \\ \alpha\{1-\cos[\phi\cos(\nu t)]\} \end{pmatrix} \tag{76}$$

If $\nu = 0$, Eq. (76) reduces to Eq. (33) as expected. A computer simulation was used to create a file of Euler rotation vectors and angular rates, generated by Eqs. (74) and (76). The proposed method was applied using the angular rate data given by Eq. (76) with $\phi = 2$ deg and $\alpha = 360$ deg/s. The coordinate transformation method was used to obtain the slew rate of the angular rate vector. The attitude was propagated, based on Eqs. (28) and (45). The resulting attitude error was found, by comparison, with the truth data given by Eq. (74). Figures 4 and 5 illustrate the angular rates and the attitude error for the case of $\nu = 0$, using a data rate of 120 Hz. As expected, the attitude error is essentially zero, with only numerical processing errors evident.

Figures 6 and 7 illustrate the results for $\nu = 20$ deg/s, with a data frequency of 120 Hz. The magnitude of the angular rate in Fig. 6 varies as expected. The slew rate was obtained using the coordinate transformation method, which assumes that the slew rate is constant. Because the slew rate changes in this case, an error is generated. Figure 7 illustrates the attitude drift rate about the z axis and the absolute magnitude of the total error at the associated time. This case shows that the proposed method can be applied to cases other than pure coning motion. However, attitude error will be generated, because the magnitude of the angular rate and the slew rate of the angular rate are not constant. Nevertheless, the error, which is in the range of 1.0×10^{-3} deg/h, is small and may be acceptable for some applications.

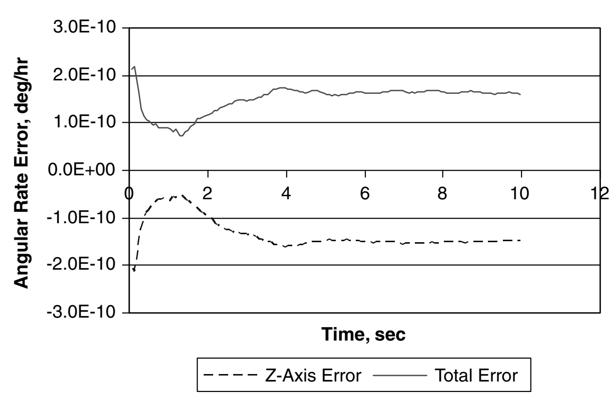

Fig. 5 Attitude error and z-axis drift rate error for the proposed method, using the angular rates in Fig. 4 with a data rate of 120 Hz.

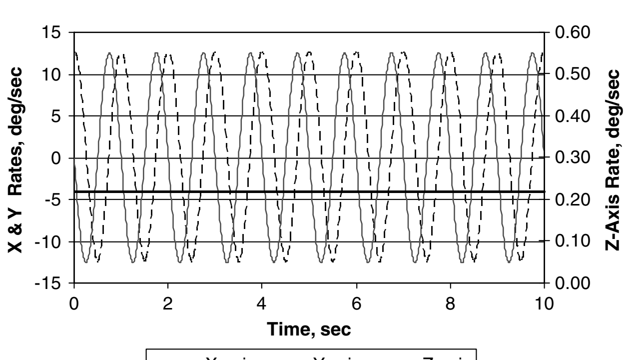

Fig. 4 Angular rates used in a case of pure coning motion.

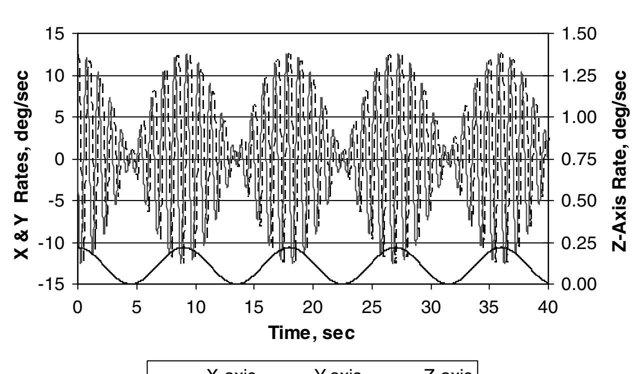

Fig. 6 Angular rates given by Eq. (75), with $\nu = 20$ deg/s.

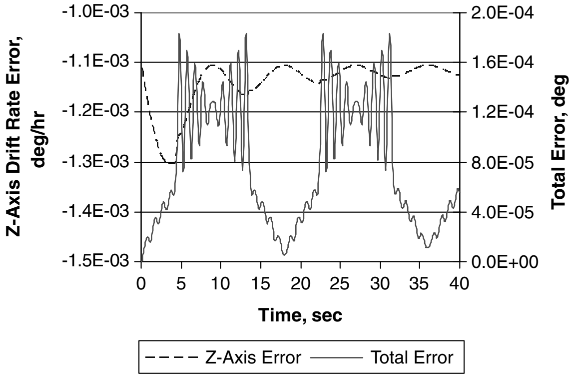

Fig. 7 Attitude error and z-axis drift rate error for the proposed method, using the angular rates in Fig. 6 with a data rate of 120 Hz.

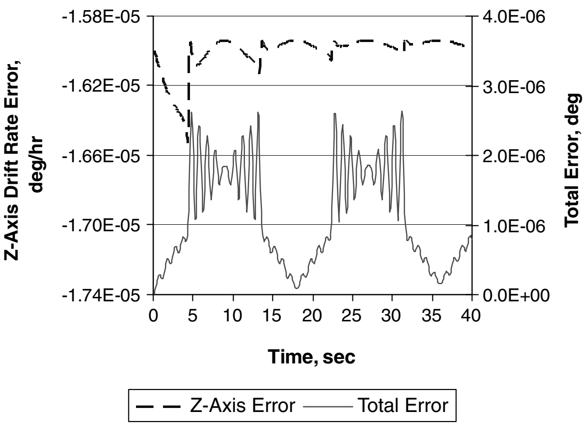

Fig. 8 Attitude error and z-axis drift rate error for the proposed method, using the angular rates in Fig. 6 with a data rate of 1000 Hz.

Figure 8 illustrates the case with a data rate of 1,000 Hz rather than the 120 Hz that is shown in Fig. 7. The higher the frequency, the lower the associated attitude error, because the angular rate magnitude and its slew rate are more nearly constant over each shorter time step. This reduces the z-axis drift rate error in Fig. 8 to roughly 1.6×10^{-5} deg/h.

Conclusions

A new method to propagate vehicle attitude is proposed that makes use of a new finding in attitude kinematics. The method uses knowledge of the slew rate of the angular rate vector to reduce attitude propagation error. It transforms the angular rate vector to a slewing reference frame, for which the attitude propagation can be performed more accurately. The attitude solution can be transformed back to the original vehicle frame as needed. This proposed method results in no error for the case of pure coning motion of the angular rate vector. It was also applied to a sequence of delta thetas that are integrals of the angular rate vector over sequential time increments. The proposed method propagated an attitude with no error for delta thetas associated with pure coning motion. It was demonstrated numerically that the new method could also be used to propagate the attitude for cases other than pure coning motion, but the attitude propagation error is no longer zero in those cases. The method provides an accurate attitude propagation algorithm for aerospace applications in both real and simulated environments.

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